

# Relative Navigation In Elliptical Orbits Using An Iterative Nonlinear Filter

James L. Garrison

*NASA Langley Research Center, Hampton, VA*

Penina Axelrad

*University of Colorado, Boulder, CO*

## BIOGRAPHY

James Garrison is a doctoral candidate in Aerospace Engineering Sciences at the University of Colorado, he holds an MS from Stanford University and a BS from Rensselaer Polytechnic Institute. He is presently employed in the Spacecraft and Sensors Branch of NASA's Langley Research Center in Hampton, VA.

Penina Axelrad is an Assistant Professor of Aerospace Engineering Sciences at the University of Colorado at Boulder. Her research and teaching are focused on GPS technology and applications. Dr. Axelrad received her Ph.D. in Aeronautics and Astronautics from Stanford University and her S.M. and S.B. from M.I.T.

## ABSTRACT

The two step filter is applied to process intersatellite radar measurements to determine the motion of one satellite relative to another in close elliptical orbits. This filter breaks a nonlinear estimation problem into two state vectors. The "first step" state is chosen so as to have a linear measurement equation. This is nonlinearly related to the "second step" state which describes the dynamics. Two different forms are used. In one, the first step state is the second step state vector augmented by the measurement equation. In the other, the first step and second step state vectors are of equal dimension. The two step filter is compared against an iterated extended Kalman filter and a Kalman filter using a change of variables. Analytical differences between the two step estimator and these conventional filters are highlighted. Special concerns for initializing the first step state covariance matrix and handling the possibility of numerically rank deficient covariance matrices are addressed. Numerical simulations are performed which show that the Two Step estimator produces a lower estimation bias under two circumstances; large apriori initial error; and small dimension observation vectors which require a longer arc of measurements to generate observability of the state.

## 1. INTRODUCTION

Several space missions have been proposed, are in

development, or are operational which require coordination of two or more satellites in a highly elliptical orbit ( $e > 0.7$ )<sup>1</sup>. Rendezvous and docking in a geostationary transfer orbit, for the purposes of satellite assembly and repair, has been considered in some advanced systems studies and has similar mission requirements<sup>2</sup>.

The relative guidance and navigation for these missions are characterized by nonlinear dynamics and frequent measurement updates of a small dimension observation vector, nonlinearly related to those states. Conventional estimation techniques applied to such problems include the extended Kalman filter (EKF), iterated extended Kalman filter (IEKF), second (and higher) order filters, and changes of variables which make the observation equation linear. All of these techniques are suboptimal in that they do not exactly minimize the global least squares cost function

$$J = \frac{1}{2} (x_o - x(-))^T P_o^{-1} (x_o - x(-)) + \frac{1}{2} \sum_{k=1}^N (y_k - h_k(x_k))^T R_k^{-1} (y_k - h_k(x_k)) \quad (1)$$

and therefore generate biases in the state estimate.

## 2. TWO STEP FILTER

The two step filter is proposed by Haupt, et. al.<sup>3</sup> and Kasdin, et. al.,<sup>4</sup> as an improved recursive solution to the nonlinear estimation problem. This filter is an exact minimization of the least squares cost function for *static* problems. In this filter, the measurement equation is expressed as a linear function of "first step" states ( $y$ ) which are nonlinearly related to the "second step" states ( $x$ ) describing the dynamics. This will allow a separable measurement equation.

$$h_k(x_o) = H_k f(x_o) \quad (2)$$

to be written as a linear function of the first step states

defined as  $y = f(x)$ . The first step state vector has to have a dimension equal to or larger than that of the second step state vector so that the second step states are observable.

In reference 3 this optimal static filter was extended to produce a sub-optimal filter for dynamic problems. The time update of the first step states is derived starting with the identity:

$$y_{k+1} = y_k + f_{k+1}(x_{k+1}) - f_k(x_k) \quad (3)$$

The expected value of the first order expansion of this expression evaluated at the  $k^{TH}$  and  $k+1^{ST}$  times approximates the first step state and covariance update as follows:

$$\bar{y}_{k+1} = \bar{y}_k + f_{k+1}(\bar{x}_{k+1}) - f_k(\bar{x}_k) \quad (4)$$

$$P_{y_{k+1}}(-) = P_{y_k}(+) + \left. \frac{\partial f}{\partial f} \right|_{k+1} P_{x_{k+1}}(-) \left. \frac{\partial f}{\partial f} \right|_{k+1}^T - \left. \frac{\partial f}{\partial f} \right|_k P_{x_k}(+) \left. \frac{\partial f}{\partial f} \right|_k^T \quad (5)$$

in which the bar (  $\bar{\phantom{x}}$  ) indicates an estimate of a vector and (-) and (+) identify the apriori and aposteriori conditions, respectively.

A standard Kalman filter measurement update is used to produce the aposteriori estimate of the first step states. This, along with the aposteriori first step covariance matrix, is used to update the second step states. When the dimension of the first step state vector is larger than that of the second step state vector, the second step state update must be performed as a numerical minimization of the least squares cost function.

$$J_{xk} = \frac{1}{2} * (\bar{y}_k(+) - f_k(x_k))^T P_{y_k}^{-1}(+) (\bar{y}_k(+) - f_k(x_k)) \quad (6)$$

The two step filter algorithm is summarized as follows:

1. Perform a linear measurement update of the first step state estimate ( $\bar{y}_k(+) \text{ )}$  and covariance ( $P_{y_k}(+) \text{ )}$  based on the  $i$ th observation.
2. Compute the second step state estimate ( $\bar{x}_k(+) \text{ )}$  by iterative minimization of  $J_{xk}$  (equation (6)). Update

the second step covariance ( $P_{xk}(+) \text{ )}$

3. Propagate the second step state estimate and covariance ( $\bar{x}_{k+1}(-) \text{ )}$  and  $P_{x_{k+1}}(-) \text{ )}$  forward to the  $k+1^{ST}$  time.
4. Compute the time update of the first step state and covariance ( $\bar{y}_{k+1}(-) \text{ )}$  and  $P_{y_{k+1}}(-) \text{ )}$  at the  $k+1^{ST}$  time using equations (4) and (5).

For more detail on the two step estimator, please consult references 3 and 4.

## 2.1 COMPARISON WITH CHANGE OF VARIABLES

Consider a special case of the two step estimator in which there are an equal number of first and second step states. In such a situation, if the function  $y_k = f_k(x_k)$  is invertible, then the second step state estimate can be solved for exactly by inverting this transformation:  $\bar{x}_k(+) = f_k^{-1}(\bar{y}_k(+))$ .

It is interesting to compare the difference between this estimator and a Kalman filter which uses a change of variables at each time step to make the measurement equation linear<sup>5</sup>. The measurement update and second step state solution for both filters are identical. The difference between them lies in the first step covariance time update of equation 5. For the coordinate transformation filter, the first step covariance update is approximated as a first order expansion.

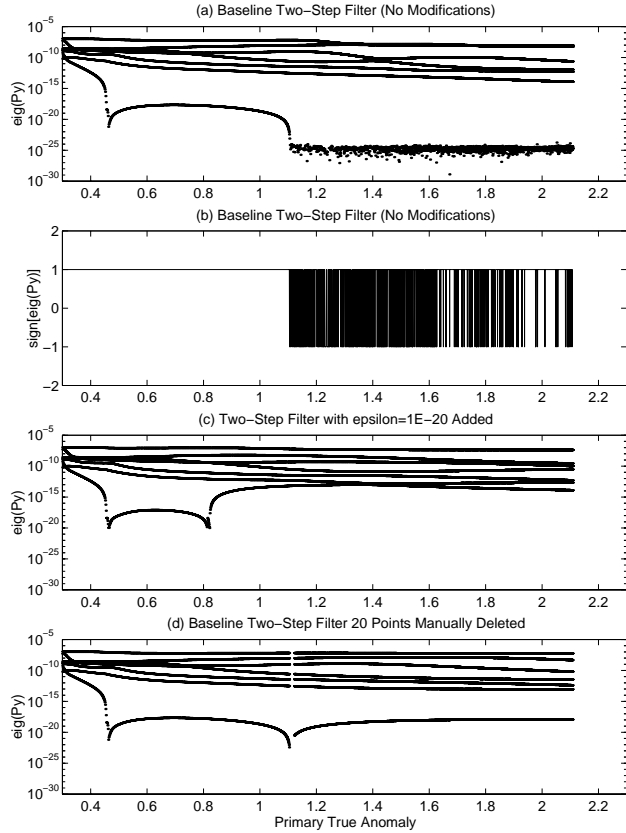
$$P_{y_{k+1}}(-) = \left. \frac{\partial f}{\partial f} \right|_{k+1} P_{x_{k+1}}(-) \left. \frac{\partial f}{\partial f} \right|_{k+1}^T \quad (7)$$

Equation (5) for the two step filter, however, only makes a first order approximation to the *change* in the first step covariance matrix. Assuming the measurements are closely spaced and that the first step covariance used to initialize the filter is a good approximation to  $E\{(\bar{y} - y)(\bar{y} - y)^T\}$  then it is expected that the two step estimator will perform better than a basic change of variables would. How well the change of variables method does in comparison to the two step filter will depend upon how close the first order approximation of  $P_y$  is to the actual first step state covariance.

## 2.2 ILL-CONDITIONED FIRST STEP COVARIANCE MATRICES

Equation (5) contains the subtraction of two positive semidefinite matrices and as such it is not always

guaranteed to generate a positive definite first step covariance matrix. In fact, in many situations, at some point this equation does produce a first step covariance matrix with a very small eigenvalue. Sometimes a small *negative* eigenvalue results, probably because of numerical reasons. A negative eigenvalue in a covariance matrix is physically meaningless and sometimes causes the second step minimization to fail or to diverge. One example of this problem is shown in Figure 1.



**Figure 1** Occurrence of an Ill-Conditioned First Step Covariance Matrix.

The plot in Figure 1 (a) shows the eigenvalues of the first step covariance matrix for a two step filter processing measurements of range rate (see section 3.3). True anomaly of the reference vehicle ("primary" as defined in section 3.1) is the independent variable. Near a true anomaly of 1.1 one of these eigenvalues gets very small. The plot in Figure 1 (b) of the sign of the smallest eigenvalue indicates that it does have negative values.

An analysis was done of the mathematical conditions for the cause and location of the small eigenvalues<sup>6</sup>. The details are beyond the scope of this paper and the interested reader is referred to reference 6. Use of a U-D

covariance factorization of the filter does reduce the effects of the problem, but does not eliminate it entirely. This is because the factored form still contains one diagonal block which is negative.

One suggested modification to the two step filter, to alleviate this problem is the addition of a small, positive diagonal matrix ( $\epsilon I$ ) onto the right hand side of equation (5).

$$P_{y_{k+1}}(-) = P_{y_k}(+) + \frac{\partial f}{\partial x} \Big|_{k+1} P_{x_{k+1}}(-) \frac{\partial f}{\partial x} \Big|_{k+1}^T \quad (8)$$

$$- \frac{\partial f}{\partial x} \Big|_k P_{x_k}(+) \frac{\partial f}{\partial x} \Big|_k^T + \epsilon I$$

The effect of this modification is shown in Figure 1 (c) which is a plot of the eigenvalues of the first step covariance matrix for the same filter, except with a term of  $\epsilon = 1E-20$  added as in equation 8. As shown, this prevents the eigenvalues from becoming negative.

Another possible modification is to skip processing measurements near the location of this problem. This is shown in Figure 1 (d) in which 20 points were manually deleted near a true anomaly of 1.1. Again, this prevents the smallest eigenvalue of  $P_y$  from becoming negative and the filter operates properly following this point.

The option of removing points for a real-time filter has the disadvantages of requiring a reliable test for ill-conditioned  $P_y$  in addition to rejecting some amount of "good" data. For a filter with an equal number of first and second step states, however, this modification was found to be more reliable than the addition of a "process noise" term in equation (8).

## 2.4 INITIALIZATION

It is important that the first step state estimate and the first step state covariance be initialized properly in order to realize the full benefits of the two step filter. The initial first step state and covariance are expressed as the expected values

$$\bar{y}_0 = E\{f_0(x_0)\} \quad (9)$$

and

$$P_{y_0} = E\{(\bar{y}_0 - y_0)(\bar{y}_0 - y_0)^T\} \quad (10)$$

These expected values are functions only of the corresponding second step state estimate and its probability distribution ( $P_x$  and  $x$ ). For a general

problem, equations (9) and (10) cannot be evaluated in closed form. A Monte-Carlo method is used in this study to compute these expected values by averaging a simulated Gaussian-distributed ensemble of second step states.

For the cases in which the initial state errors are assumed to be Gaussian and un-correlated, it is also possible as well as practical to numerically integrate the expected value for each element of the first step state estimate and covariance.

$$E\{g(x)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(x) p(x_1) p(x_2) \dots p(x_2) dx_1 dx_2 \dots dx_3 \quad (11)$$

Numerical integration is practical for all of the formulations of the first step states described in section 3.3 because the velocity terms appear only linearly. This, combined with the assumption of uncorrelated initial conditions, allows factoring the expected values of the velocity terms outside of the integral, leaving only three as opposed to six numerical integrations.

It may not always be possible to assume that the six initial states are uncoupled. For those cases in which there are significant off diagonal elements of  $Px$  then a Monte Carlo method using a coordinate transformation is recommended.

Both methods were applied to the same problem for validation and it is found that they generally agree with each other to within 10% when  $2E6$  Monte Carlo samples are used and the numerical integrations are performed over intervals from  $-5\sigma$  to  $+5\sigma$ . Some very small elements of  $P_y$ , however did show larger relative errors in that number of Monte Carlo simulations. These errors were reduced when the Monte Carlo simulations were done using the scaled first step states defined in section 3.2.

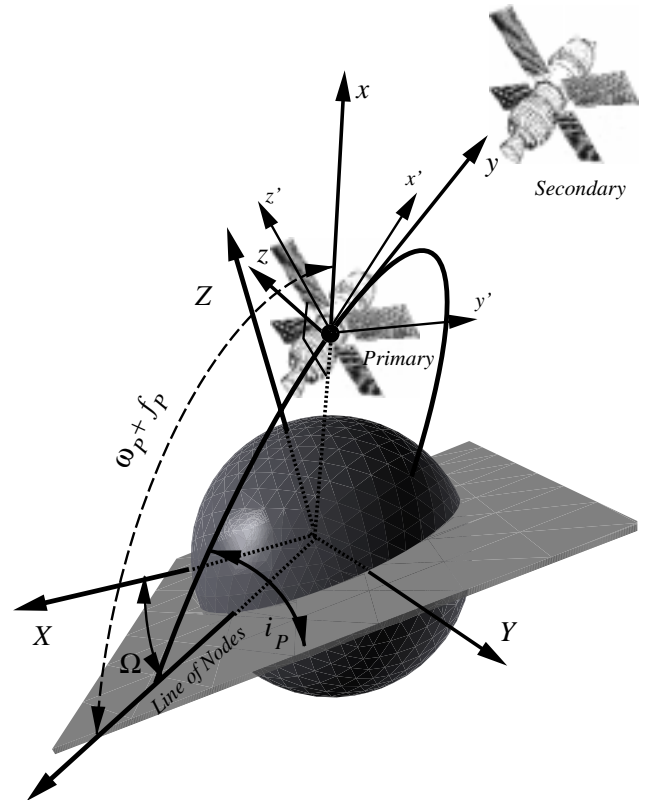
### 3. RELATIVE NAVIGATION

#### 3.1 BASIC ASSUMPTIONS

The two spacecraft are identified as the “primary” and the “secondary” and all relative dynamics described in terms of motion of the secondary with respect to the location of the primary. The navigation sensors are located on the primary vehicle and provide range, range rate, elevation and azimuth of the secondary vehicle with respect to the primary.

Three different coordinate systems are used to describe this motion as shown in Figure 2. The Earth centered inertial (ECI) system is defined at the center of mass of

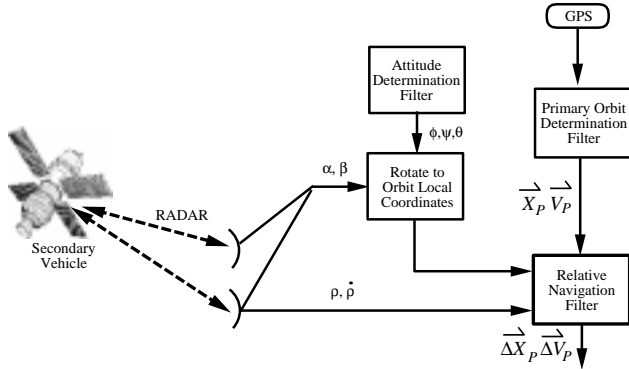
the Earth with the X axis along the line of nodes of the primary vehicle orbit and the Z axis in the direction of the Earth’s angular momentum vector. The Y axis is in the equatorial plane completing a right-handed system. The right ascension of the ascending node of the primary is therefore zero ( $\Omega_p \equiv 0$ ). This reference frame is assumed to be inertial in this study and it is used to integrate the dynamics between measurements. A local system is defined in which x is along the radius vector, z is perpendicular to the orbit plane and y completes the right hand triad which forces it to be oriented in the direction of orbital motion. Raw measurement data from the sensors are referenced to some body-fixed coordinate frame, shown symbolically as  $\{x', y', z'\}$ .



**Figure 2** Coordinate System Definitions

Relative navigation is performed using only intersatellite ranging measurements and estimating only relative states. Solution of the complete navigation problem, however, requires determination of not only the six relative states but the orbit of the primary vehicle as well. Attitude determination of the primary may or may not be a

required output, depending upon the application and the control requirements on this vehicle. In any case, attitude knowledge is necessary for the proper incorporation of the elevation and azimuth measurements. In this study, perfect attitude and orbit knowledge for the primary vehicle is assumed. This assumption allows the elevation and azimuth to be expressed in the *local* reference system as opposed to the *body* reference. The relationship between the relative navigation filter to be developed and the external orbit and attitude determination filters is illustrated in Figure 3.



**Figure 3** Relationship between Relative Navigation, Orbit Determination, and Attitude Determination Filters

### 3.2 DYNAMIC MODEL

The relative navigation filter estimates a six element state vector consisting of the differences in inertial position and velocity between the primary vehicle and the secondary vehicle in the ECI reference frame of Figure 2. The relative position states are scaled by a reference semimajor axis value and the relative velocities are scaled by a reference circular velocity to reduce potential numerical problems. The corresponding components of the second step states in all of the filters used in this study are identified by subscripts:  $x = [x_1, x_2, \dots, x_6]^T$

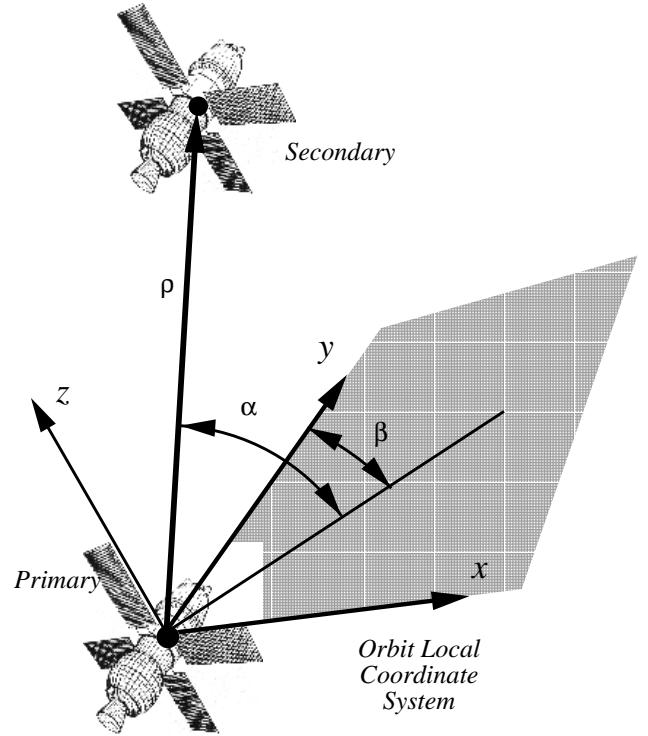
$$\begin{aligned} x_1 &= \frac{X_S - X_P}{a_p} & x_2 &= \frac{Y_S - Y_P}{a_p} & x_3 &= \frac{Z_S - Z_P}{a_p} \\ x_4 &= \frac{U_P - U_S}{\sqrt{\frac{\mu}{a_p}}} & x_5 &= \frac{V_P - V_S}{\sqrt{\frac{\mu}{a_p}}} & x_6 &= \frac{W_P - W_S}{\sqrt{\frac{\mu}{a_p}}} \end{aligned}$$

For this study it is assumed that the orbit of the target is known to a high precision and the orbit elements of this vehicle will be used as constant parameters of the filter. The filter dynamic model further assumes that both the primary and the secondary vehicles are in two-body orbits.

The primary orbit is therefore defined by the constant Keplerian orbit elements; semimajor axis ( $a_p$ ); eccentricity ( $e_p$ ); argument of perigee ( $\varpi_p$ ); and inclination ( $i_p$ ). True anomaly of the primary ( $f_p$ ) is used as the independent variable. This eliminates the need to separately propagate the primary vehicle orbit

### 3.3 OBSERVATIONS AND FIRST STEP STATES

The first set of measurements considered are those of range, rate-rate, elevation and azimuth. This set of measurements makes the state estimate very observable and the filter is less dependent upon the dynamic model to incorporate measurements over a long arc of data. This set of observations eliminates problems with weakly observable out of plane errors for nearly coplanar orbits. It will also prevent some possible ambiguities in which the iterations converge to the wrong orbit. Elevation and azimuth are expressed in the local reference frame as illustrated in Figure 4.



**Figure 4** Definition of Azimuth and Elevation in the local coordinate system.

With this set of measurements defined above, the nonlinear observation equation is

$$h_x(x) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_1 x_4 + x_2 x_5 + x_3 x_6 \\ \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \alpha \\ \beta \end{bmatrix} \quad (12)$$

in which

$$\alpha = \frac{-x_2 \sin i_p + x_3 \cos i_p}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \quad (13)$$

$$\beta = \frac{[x_1 \cos(\varpi_p + f_p) + x_2 \cos i_p \sin(\varpi_p + f_p) + x_3 \sin i_p \sin(\varpi_p + f_p)]}{(x_1^2 + x_2^2 + x_3^2)^{-1/2}} \quad (14)$$

In one application, the first step state vector is formed by augmenting the second step states with the measurement equation.

$$f(x) = [h(x), x]^T \quad (15)$$

To construct a two step filter with first and second step states that have the same dimension, the following first step state vector is defined:

$$f(x) = \begin{bmatrix} \frac{x_1 x_4}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \frac{x_2 x_5}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \frac{x_3 x_6}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \alpha \\ \beta \end{bmatrix} \quad (16)$$

As a comparison to the previous filter which used a large-dimension observation vector, a filter processing a single scalar measurement of range rate is also considered. Both a state augmented form of the first step state as well as the first step state vector defined in equation (16) of equal dimension to the second step state vector are used.

#### 4.0 NUMERICAL SIMULATION

Two scenarios are simulated which best illustrate the advantages of the two step estimator. The first scenario uses a large dimension measurement vector giving good observability of the state vector, but the filter is initialized with a large a priori error.

The second scenario uses the filter to process scalar measurements and as such is more dependent upon the filter to make the state observable. It is simulated with an ensemble of initial conditions.

#### 4.1 First Scenario: Large Dimension Observation Vector and Large Apriori Error

This scenario processes measurements of range, range rate, elevation and azimuth. For one application of the two step filter, the first step states are defined by augmenting the second step state vector with the measurement equation as in equation (15). Another two step filter is derived using a first step state defined in equation (16) so that it has an equal dimension to the second step state vector. A coordinate transformation filter is similarly applied using those same first step states, in which the nonlinear change of variables defined by equation (16) is used at each time step. An IEKF is also simulated for comparison.

The reference conditions and uncertainty are based on an ESA study of rendezvous and docking in geostationary transfer orbits in reference 2. The specific numbers are listed in Table 1.

**Table 1** First Scenario: Filter A Priori Orbit Elements.

Primary	Semimajor Axis	24371 km
	Eccentricity	0.7301
	Argument of Perigee	180 deg.
	Inclination	8 deg.
	True Anomaly at Start of Simulation	0 deg.
Secondary	Xs-Xp	0.4 km
	Ys-Yp	-97.7 km
	Zs-Zp	-13.7 km
	Us-Up	88.8 m/sec
	Vs-Vp	0.7 m/sec
	Ws-Wp	0.1 m/sec

Initially, the primary and secondary are assumed to lie in the same orbit except for a separation in true anomaly. The uncertainties stated in that study are derived from predicted launch vehicle dispersions following a period of ground-based tracking of the two vehicles independently.

The simulations are run for 3200 seconds, processing measurements once a second starting with the primary vehicle at perigee. A value of  $\epsilon = 1E-18$  is added as in equation (8) and this prevented any failures resulting from ill-conditioned  $P_y$  over the ensemble of cases ran in this simulation. For the six state filter, however, this value of  $\epsilon$

does not completely prevent an ill conditioned  $P_y$  from causing an increase in error. It is found to be very difficult to properly set  $\epsilon$  for a filter with equal dimensioned states. One possible reason for this is that, for a filter with an equal number of first and second step states, an ill-conditioned first step covariance would only occur for a truly (numerically) rank deficient partial derivative matrix. When such a near-singularity occurs, it is not expected that simply adding a small positive diagonal matrix onto it would improve its condition number. (If  $A$  is ill-conditioned, then so is  $A + \epsilon I$  for small  $\epsilon$ .) In the next section, this problem is worse and it is necessary to skip points in order to avoid severe numerical problems as a result of poorly conditioned  $P_y$ .

A total of 30 simulated trajectories were generated in which each set of simulated noisy data was processed by all four filters. The same tolerances are used for each of the filters that require an iterative solution. Iterations are stopped when the change in state is less than  $\max(|\delta x| / \|x\|) \leq 1E-12$  after a minimum of 10 iterations have occurred. A maximum of 500 iterations is allowed, to prevent infinite loops in the event of divergent solutions.

All of the estimators considered are expected to do well. The two step filter, however, demonstrates better performance when the set of simulated runs are started from a large initial error. The initial error used in all of these simulate trajectories is given in Table 2. The initial diagonals of the second step covariance matrix (derived from the ESA study in reference 2) is given in Table 3. Elements of the first step covariance matrix were obtained by Monte Carlo simulation.

**Table 2** First Scenario: A Priori Errors

(In ECI coordinates as shown on Figure 2)

Xs-Xp Error	10 km	$2.8 \sqrt{P_{11}}$
Ys-Yp Error	-6 km	$0.5 \sqrt{P_{22}}$
Zs-Zp Error	1 km	$0.6 \sqrt{P_{33}}$
Us-Up Error	20 m/sec	$1.0 \sqrt{P_{44}}$
Vs-Vp Error	8 m/sec	$0.8 \sqrt{P_{55}}$
Ws-Wp Error	1 m/sec	$0.7 \sqrt{P_{66}}$

This has a root-sum-square magnitude of  $3.2 \sigma$ .

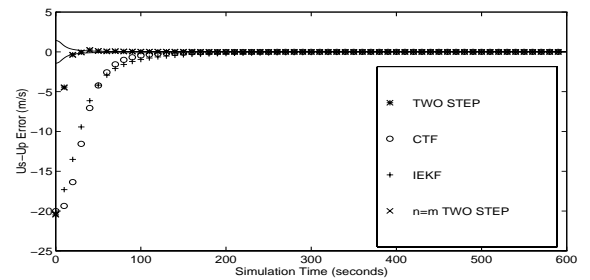
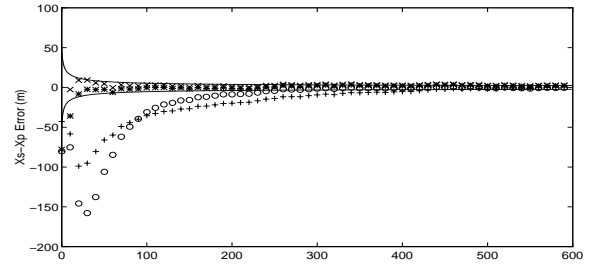
Numerical results showing the mean position errors for the four different filters over the first 600 seconds of the simulation are given in Figure 5. This figure also plots the standard deviation of the mean  $\sigma / \sqrt{N}^7$  in which  $N$  is

the number of measurements used in the average and  $\sigma$  is the standard error predicted by the filter for each particular state element. Most of the two step filter simulation state history lies within these bounds and is thus considered to be "unbiased" within the statistical significance of this simulation. Whereas these plots demonstrate that all four filters converge for a large initial error and perform rather well, the two step filter in both forms is better at removing large initial errors. This shows that the steady state covariance propagation and update of the two step filter is approximately the same as that of a Kalman filter operating on the linearized model. This assumption was important in the analysis of the location of rank deficient  $P_y$  in reference 6, allowing examination of the covariance propagation independent of the state estimate.

**Table 3** First Scenario: Second Step Covariance

(In ECI coordinates as shown on Figure 2)

$P_{XX}$	1.3E7 m <sup>2</sup>	$P_{UU}$	442 m <sup>2</sup> /sec <sup>2</sup>
$P_{YY}$	1.6E8 m <sup>2</sup>	$P_{VV}$	100 m <sup>2</sup> /sec <sup>2</sup>
$P_{ZZ}$	3.2E6 m <sup>2</sup>	$P_{WW}$	2 m <sup>2</sup> /sec <sup>2</sup>

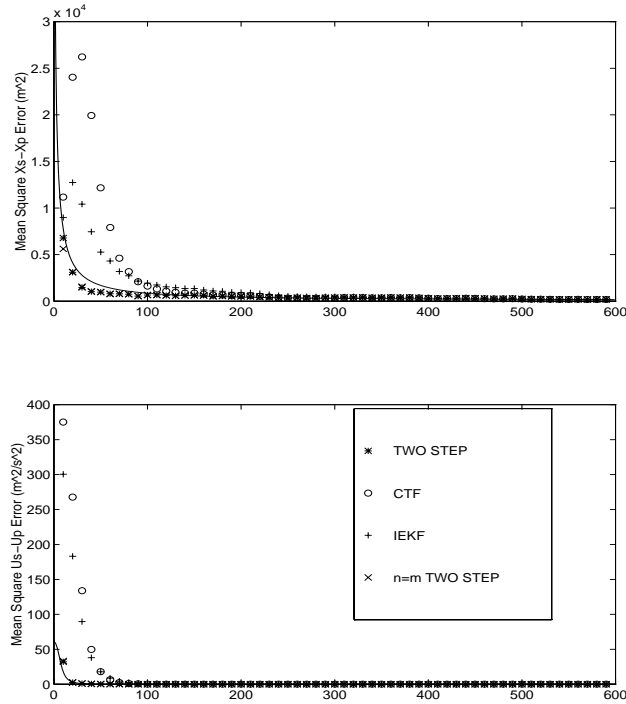


**Figure 5** First Scenario: Mean State Error

Comparison of the respective curves in Figure 5 demonstrates that the two step filter provides a lower error than the coordinate transformed filter using the same set of states. The differences are most significant for the velocity states. This can be explained by the fact that three of the four observations; range, elevation, and azimuth are

direct measurements of position. Range rate is the only direct measurement of velocity. The velocity state estimate is therefore more dependent upon the filter to propagate it from one measurement to the next.

The comparison between the predicted covariance and the mean squared error for the filters is given in Figure 6. Predicted covariance is indicated by the solid line for the two step filter and the dotted line for the IEKF. That figure confirms the position and velocity plots in Figure 5. Again, note that the improvement in velocity error is larger than that in position error.



**Figure 6** First Scenario: Mean Square Error

It can be concluded from these simulations that, when the measurement set is large and available frequently the advantage of the two step estimator is in the removal of large initial errors which fall outside of the covariance bounds assumed for the filter.

The mean execution times for the complete 3200 second simulation of each filter are compared in Table 4. Numbers in this table are all normalized with the ten state two step filter. The lower numbers for the coordinate transformation and the six state two step filter are expected when one considers that in those filters the second step states are obtained by a closed form equation

and no iteration is required. The higher numbers for the IEKF are due to the fact that a U-D factored formulation is used for that filter. That algorithm requires scalar measurements<sup>8</sup>. Hence each of the four observations must be processed individually, requiring four times the number of iterations as in the two step filter. Each iteration in the IEKF involves fewer operations than each in the two-step filter, however.

**Table 4** Mean Execution Time for the First Scenario.

Two Step Filter	1 (defn.)
IEKF	1.51
Coordinate Change Filter	0.30
Two Step Filter with $n=m=6$	0.37

#### 4.2 Second Scenario: Small Dimension Observation With Ensemble of Apriori Error

In the second scenario only range rate measurements are used. For this case, an ensemble of initial errors are generated by a Monte-Carlo method using the initial second step state covariance and assuming that the second step states are initially uncorrelated. Both forms of the two step estimator; the state augmented one and the  $n = m$  filter using equation (16). as well as the IEKF and coordinate transformation filters are simulated.

The filter in this scenario will be more heavily relied upon to combine the measurements and make the state observable than the one in the first scenario. The problem of state observability is much worse for the example mission in the first scenario in which both vehicles are in the same orbital plane. In that case the out of plane motion is very weakly observable. Some form of angle measurements is necessary for a robust navigation system in that orbital configuration.

The objective at this time is to demonstrate the difference in estimation accuracy between the two step filters and the IEKF and coordinate transformation filters. As such, an example orbit case which includes some out of plane motion as well as some radial motion is devised. This is formed by adding inclination and eccentricity differences to the reference orbits in Table 1. A smaller initial covariance is also used in these simulations. The scenario was started at primary true anomaly of -60 deg. and each simulation is run for 6000 seconds. A total of 40 Monte Carlo simulations are performed. The initial states are selected by adding randomly generated errors to the reference values in Table 5. These errors are zero mean, uncorrelated Gaussian random numbers with statistics

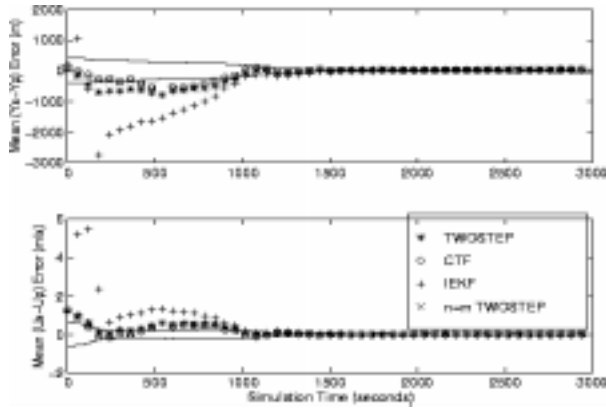
given by the covariance matrix terms in Table 5. Again, the Monte-Carlo integration method is used to generate the apriori first step state estimate and covariance from the data in Table 5.

**Table 5** Second Scenario: Second Step Covariance

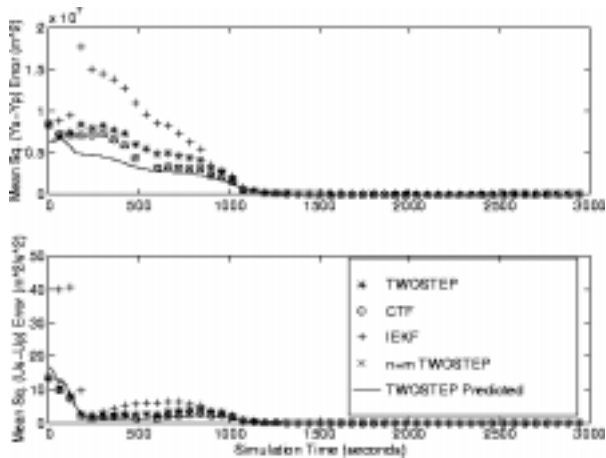
(In ECI coordinates as shown on Figure 2)

$P_{XX}$	1E6 m <sup>2</sup>	$P_{UU}$	17.6 m <sup>2</sup> /sec <sup>2</sup>
$P_{YY}$	1.7E7 m <sup>2</sup>	$P_{VV}$	1.4 m <sup>2</sup> /sec <sup>2</sup>
$P_{ZZ}$	3.2E5 m <sup>2</sup>	$P_{WW}$	0.03 m <sup>2</sup> /sec <sup>2</sup>

Figure 7 plots the mean error for these simulations, along with the standard deviation of the mean. Figure 8 is a plot of the mean squared error and the filter predicted variance. None of the ensemble of 40 Monte Carlo runs diverge. When a larger initial covariance, comparable in magnitude to that used in the range-rate-angles case is used, there are divergent runs.



**Figure 7** Second Scenario: Mean State Error



**Figure 8** Second Scenario: Mean Square Error

For the case of equal dimensioned first and second step states, the addition of a small  $\epsilon I$  to the first step state covariance propagation in equation (8) is not always able to mitigate the difficulties associated with rank deficiency of the partial derivative matrix itself. The solution in this simulation is to compute the rank of this matrix at each time step using a tolerance of  $1E - 3$ . If the rank dropped at anytime then the next 20 seconds of measurements are skipped.

The state estimation error using the coordinate transformation filter is comparable to that for the seven state two step filter and both the coordinate change filter and the six state filter have results which are very similar. This appears to contradict the earlier analysis which indicated that the two step filter with an equal dimensioned first and second step state vector would produce a better state estimate because it makes a smaller approximation to the covariance update between the first and second step states. For this specific case, however, the first step state covariance is much closer to the first order approximation than what is used in the first scenario. This is shown by comparing the norm of the difference between the initial  $P_y$  and the first order approximation, divided element-wise by  $P_y$ .

$$\left\| \left( \left( \frac{\partial f}{\partial x} P_x \frac{\partial f^T}{\partial z} \right)_{i,j} - P_{y_{i,j}} \right) / P_{y_{i,j}} \right\|$$

The norm of this (scaled) matrix is 16 times larger for the example in the first scenario than it is for the example in the second scenario. Hence the error made in approximating the first step covariance by the first order expression is larger for the first scenario than it is for the second.

The Monte Carlo simulations presented in this chapter demonstrate that the two step filter provides a better mean square state estimate than the IEKF and the coordinate transformation filter under two circumstances. The first is when the initial error is very large as compared to the expected state uncertainty and the second is with systems that have a small dimension observation vector and as such are dependent upon the filter combining measurements over time to make the state observable.

## 5.0 CONCLUSIONS

The findings of this study indicate that the two step estimator would be a viable on-board relative navigation filter. This filter offers better state estimation accuracy for cases in which the initial state error exceeds its predicted bounds as well as situations in which a small measurement

set is used. The latter case is particularly interesting in that this improvement in filtering may allow the use of fewer and simpler navigation sensors thereby reducing mission cost. The processing time requirements for the two step filter using an iterative solution were comparable to those of the IEKF and approximately 3 times longer than those of the coordinate transformation based filter. Special problems in using the two step estimator such as initialization of the first step covariance and the possibility of numerically rank-deficient covariance matrices were solved. Further study is needed, however, into the robustness of this filter, especially the rank deficiency problem. One possible implementation is to use the two step filter for the initial acquisition of the target vehicle and then to switch to a more conventional IEKF after the state error has decreased.

---

1979, pp 392-399.

## 6.0 REFERENCES

- <sup>1</sup> Telephone conversation with Dr. Stephen A. Curtis, Planetary Magnetosphere Branch, Goddard Space Flight Center, January 11, 1996.
- <sup>2</sup> *System Feasibility Study on Rendezvous and Docking in Transfer Orbit*, Matra Espace, February, 1983.
- <sup>3</sup> Haupt, G. T., Kasdin, N. J., Keiser, G. M., and Parkinson, B. W., "Optimal Recursive Iterative Algorithm for Discrete Nonlinear Least Squares Estimation," *Journal of Guidance, Control and Dynamics*, Vol. 19, No. 3, May-June 1996, pp 643-649.
- <sup>4</sup> Kasdin, N. J., and Haupt, G. T., "Second Order Correction and Numerical Considerations for the Two-Step Optimal Estimator," *Journal of Guidance, Control and Dynamics*, Vol. 20, No. 2, March-April 1997, pp362-369.
- <sup>5</sup> Mehra, R. K., "A Comparison of Several Nonlinear Filters for Reentry Vehicle Tracking," *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 4, August, 1971, pp 307-319.
- <sup>6</sup> Garrison, J. L., Axelrad, P., and Kasdin, N. J., "On the Possibility of Ill-Conditioned Covariance Matrices in the First-Order Two-Step Estimator," Presented at the *AIAA/AAS Space Flight Mechanics Meeting*, Huntsville, AL, February 10-12, 1997, paper AAS 97-195.
- <sup>7</sup> Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, Third Edition, McGraw-Hill, Inc., 1982, p246.
- <sup>8</sup> Maybeck, P. S., *Stochastic Models, Estimation and Control*, Volume 2., Academic Press, San Diego, CA,